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# Orthogonality condition for a multi-span beam, and its application to transient vibration of a two-span beam

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**Abstract**

This paper treats the orthogonality condition for a multi-span beam, and its application to forced (transient) vibration of a two-span beam. The beam is modeled as a Bernoulli–Euler beam. The boundary conditions for the particular case of two-span beam are clamped–pinned–pinned. An exact closed-form solution is obtained for this problem. Even though there has been an enormous amount of work on beam vibration, most of the studies are conducted on a single-span beam. There are some studies on the multi-span beam vibration. However, their treatment is rather specialized in terms of the applied loading and the initial conditions. None of the studies in the past treats an exact solution for a forced (transient) vibration of a general two-span beam with arbitrary initial conditions and arbitrary forcing functions. Therefore, the solution obtained in this paper is new. The key development in the solution is the orthogonality condition for a multi-span beam. The method of solution developed in this paper establishes a general methodology for the forced (transient) vibration of a multi-span beam. The closed-form solution obtained in this paper can be used as a benchmark solution for the transient vibration of a two-span beam.


## 1. Introduction

There has been an enormous amount of study on the vibration of beams in the literature. However, most of these studies are conducted on a single-span beam [1–8]. There are studies on multi-span beam vibration [5–19]. However, most of them focus on either numerical approximate solutions of free vibration as well as forced vibration, or exact analysis for natural frequencies of free vibration. There are some exact analyses of forced vibration [9,10,14,18,19]. However, they are for a special case of equal span and uniform cross-section for the entire span [9,10], three-span beam of equal mass density [14], a continuous beam with an infinite number of identical spans [18], or a continuous beam under moving loads with zero initial conditions [19]. The most general analytical solution for forced vibration of a multi-span beam in the literature seems to be the one given by Dugush and Eisenberger [19]. It is based on the polynomial representation of the mode shape, and the solution is not closed form. Even though the treatment of the eigenfunctions (mode shapes) in Ref. [19] is fairly general, it may not be called exact, since the eigenvalues are based on an approximate eigenvalue

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problem due to the inevitable truncation of the infinite series used for the mode shape. In other words, no matter how accurately the characteristic equation for the eigenvalue problem is solved, the eigenvalues are still approximate, since the characteristic equation is based on the truncated polynomial power series representation of the mode shape instead of an exact representation based on the transcendental functions (sin, cos, sinh, and cosh). Also, the determination of the expansion coefficients for the mode shape of the entire span (from the first span to the last span) was not made explicit in Ref. [19]. In the treatment of the forced vibration in their paper [19], Eq. (29) of Ref. [19] was not exactly correct unless $EI(x)$ is constant.

In the field of multi-span beam vibration, the overwhelming majority of the literature [7–19] discusses beam vibration under a moving load. Accordingly, the analysis of the vibration is rather specialized in terms of the applied loading and the initial conditions. As far as the mathematics is concerned, the orthogonality condition is the key to solve the forced (transient) vibration of the multi-span beam with arbitrary boundary conditions, arbitrary initial conditions, and arbitrary forcing functions. Relatively few references explicitly discuss the orthogonality condition for the multi-span beam [14,16,19]. However, none of the references provide the mathematical proof of the orthogonality condition for the general multi-span beam. Judging from the literature, it seems that the orthogonality condition for the multi-span beam is taken for granted. Fryba [8], for example, derives many equations assuming the orthogonality of the normal modes for a single-span beam and a rectangular plate, but he never proves the orthogonality (see Eqs. (6.8), (12.21), (15.30) in Ref. [8]). Furthermore, his treatment of a multi-span beam is from an earlier literature, which only covers an equal span beam, not a general multi-span beam. In his book, Fryba even states (in Section 12.2.1 in Ref. [8]) that computing the forced vibration is simple, but what is difficult is the determination of free vibration and normal modes. This statement is true assuming that the orthogonality of the normal modes exists. Many authors cite Fryba's book [8] and apparently follow his approach. However, the orthogonality of the normal modes should not be taken for granted for an arbitrary structure, as Blevins discusses the non-existence of the general orthogonality principle in the case of mode shapes of vibration of thin plates (Section 11-1 in Ref. [5]). Blevins even states that a general closed-form solution does not exist for vibration of a rectangular plate with various elementary boundary conditions on each of the four edges (Section 11-3 in Ref. [5]). The existence of the orthogonality condition strongly depends on the boundary conditions associated with the boundary value problem (e.g., vibration problem). Consequently, the orthogonality condition has to be established for each type of boundary value problems. It should be noted here that the usual Sturm–Liouville theory for the self-adjoint differential operator of the second order is applicable only to the single-span beam vibration [20–22]. Therefore, the treatment of the orthogonality condition in this paper can be considered as an application of the extension of the Sturm–Liouville theory to the multi-span beam.

One of the major results of this paper is the mathematical proof of the orthogonality condition for the multi-span beam of variable cross-section. Since this is a general orthogonality condition for the multi-span beam, it can be applied to a forced (transient) vibration of a multi-span beam with variable cross-section. However, in this paper, as a specific application of this general orthogonality condition, we have obtained an exact closed-form solution for the forced (transient) vibration of a two-span beam, and examined the numerical results of this solution. Even though only the transient vibration of a two-span beam is discussed as an application of the general orthogonality condition, this paper nonetheless establishes a method of solution for the forced (transient) vibration of a multi-span beam of variable cross-section.

In Section 2, the statement of the problem is given. The orthogonality of the eigenfunctions for a multi-span beam with variable cross-section is established in Section 3. The mathematical formulation for a two-span beam is given in Section 4, the determination of eigenfunctions is given in Sections 5 and 6, applications to the transient vibration of a beam with a base motion are given in Sections 7 and 8, the numerical results and discussion are given in Section 9, and finally the conclusion is given in Section 10.

## 2. Problem statement

The schematic view of a two-span beam is shown in Fig. 1. The left end is clamped, and the mid-point and the right end are simply supported. Our objective is to determine the dynamic behavior of this two-span beam. The beam is modeled as a Bernoulli–Euler beam. The governing equation for the Bernoulli–Euler beam in

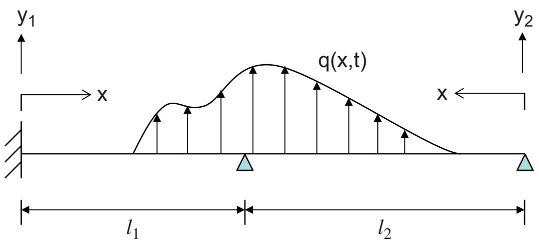


Fig. 1. Schematic view of the two-span beam vibration with two-coordinates systems.

each span is given by

$$E_1I_1\frac{\partial^4 y_1}{\partial x^4}+\rho_1A_1\frac{\partial^2 y_1}{\partial t^2}=q_1(x,t)\quad(0<x<l_1),$$
$$E_2I_2\frac{\partial^4 y_2}{\partial x^4}+\rho_2A_2\frac{\partial^2 y_2}{\partial t^2}=q_2(x,t)\quad(0<x<l_2),\tag{1}$$

where the positive direction of the spatial coordinate ($x$) is defined in the direction to the right for the left span ($0<x<l_1$), and it is defined in the direction to the left for the right span ($0<x<l_2$). This notational convention is adopted here to simplify the algebra. The boundary conditions are given by

$$\begin{aligned}&y_1(0,t)=0,\quad y_1'(0,t)=0,\\&y_1(l_1,t)=0,\quad y_2(l_2,t)=0,\quad y_1'(l_1,t)=-y_2'(l_2,t),\quad E_1I_1y_1''(l_1,t)=E_2I_2y_2''(l_2,t),\\&y_2(0,t)=0,\quad y_2''(0,t)=0.\end{aligned}\tag{2}$$

The initial conditions are given by

$$y_1(x,0)=f_1(x),\quad \frac{\partial y_1}{\partial t}(x,0)=g_1(x)\qquad(0<x<l_1),$$
$$y_2(x,0)=f_2(x),\quad \frac{\partial y_2}{\partial t}(x,0)=g_2(x)\qquad(0<x<l_2).\tag{3}$$

Our goal is to obtain the solution for Eq. (1) together with Eqs. (2) and (3).

## 3. Orthogonality of the eigenfunctions for a multi-span beam

In order to solve the boundary value problem defined in Eqs. (1)–(3), we need to establish the orthogonality of the eigenfunctions for a two-span beam. Since the mathematical treatment of two-span beam and multi-span beam is essentially the same, we will establish the orthogonality of the eigenfunctions for a multi-span beam with variable cross-section so that a forced (transient) vibration of the multi-span beam can be solved to utmost generality. We proceed as follows. First the proof is given for the orthogonality of the eigenfunctions for a two-span beam. Then by extending the proof, the orthogonality of the eigenfunctions for a multi-span beam is established.

In order to prove the orthogonality of the eigenfunctions for a two-span beam, let us consider the following two eigenvalue problems.

**Problem 1.**

$$L_1(u_{1m})-\omega_m^2r_1(x)u_{1m}=0,\quad 0\leqslant x\leqslant l_1,$$
$$L_2(u_{2m})-\omega_m^2r_2(x)u_{2m}=0,\quad 0\leqslant x\leqslant l_2,\tag{4}$$

where

$$L_1=\frac{d^2}{dx^2}\left(p_1(x)\frac{d^2}{dx^2}\right),\quad L_2=\frac{d^2}{dx^2}\left(p_2(x)\frac{d^2}{dx^2}\right),\tag{5}$$

$$p_i(x) = E_i(x)I_i(x), \quad r_i(x) = \rho_i(x)A_i(x) \qquad (i = 1, 2). \tag{6}$$

The boundary conditions for $u_{1m}$ and $u_{2m}$ are given by

$$\begin{aligned}
&u_{1m}(0) = 0, \quad u'_{1m}(0) = 0,\\
&u_{1m}(l_1) = 0, \quad u_{2m}(l_2) = 0, \quad u'_{1m}(l_1) = -u'_{2m}(l_2), \quad p_1(l_1)u''_{1m}(l_1) = p_2(l_2)u''_{2m}(l_2),\\
&u_{2m}(0) = 0, \quad u''_{2m}(0) = 0.
\end{aligned} \tag{7}$$

**Problem 2.**

$$\begin{aligned}
&L_1(u_{1n}) - \omega_n^2 r_1(x)u_{1n} = 0, \quad 0 \leqslant x \leqslant l_1,\\
&L_2(u_{2n}) - \omega_n^2 r_2(x)u_{2n} = 0, \quad 0 \leqslant x \leqslant l_2.
\end{aligned} \tag{8}$$

The boundary conditions for $u_{1n}$ and $u_{2n}$ are given by

$$\begin{aligned}
&u_{1n}(0) = 0, \quad u'_{1n}(0) = 0,\\
&u_{1n}(l_1) = 0, \quad u_{2n}(l_2) = 0, \quad u'_{1n}(l_1) = -u'_{2n}(l_2), \quad p_1(l_1)u''_{1n}(l_1) = p_2(l_2)u''_{2n}(l_2),\\
&u_{2n}(0) = 0, \quad u''_{2n}(0) = 0.
\end{aligned} \tag{9}$$

It should be noted that the above two problems are the same eigenvalue problem except that the solution for each problem can be different. It should be also noted that each eigenvalue problem is a pair of ordinary differential equations, which are coupled in the boundary conditions. By using integration by parts, it can be shown from Eqs. (7) and (9) that

$$\begin{aligned}
&\int_0^{l_1} [u_{1m}(p_1(x)u''_{1n})'' - u_{1n}(p_1(x)u''_{1m})''] \, \mathrm{d}x + \int_0^{l_2} [u_{2m}(p_2(x)u''_{2n})'' - u_{2n}(p_2(x)u''_{2m})''] \, \mathrm{d}x\\
&\quad = p_1(0)u''_{1n}(0)u'_{1m}(0) - p_1(0)u''_{1m}(0)u'_{1n}(0) - (p_1(x)u''_{1n})'|_{x=0}u_{1m}(0) + (p_1(x)u''_{1m})'|_{x=0}u_{1n}(0)\\
&\qquad + p_2(0)u''_{2n}(0)u'_{2m}(0) - p_2(0)u''_{2m}(0)u'_{2n}(0) - (p_2(x)u''_{2n})'|_{x=0}u_{2m}(0) + (p_2(x)u''_{2m})'|_{x=0}u_{2n}(0).
\end{aligned} \tag{10}$$

It can be seen from Eq. (10) that all the terms associated with the intermediate support are eliminated thanks to the intermediate boundary conditions given by the middle four expressions in both Eqs. (7) and (9). It should be mentioned that Eq. (10) is valid for arbitrary end boundary conditions. If the particular end boundary conditions that are given in the first two and the last two expressions of both Eqs. (7) and (9) are substituted into Eq. (10), the following is obtained:

$$\int_0^{l_1} [u_{1m}(p_1(x)u''_{1n})'' - u_{1n}(p_1(x)u''_{1m})''] \, \mathrm{d}x + \int_0^{l_2} [u_{2m}(p_2(x)u''_{2n})'' - u_{2n}(p_2(x)u''_{2m})''] \, \mathrm{d}x = 0. \tag{11}$$

It can be easily seen from Eq. (10) that Eq. (11) would have been obtained from any other common end boundary conditions. From Eqs. (4), (8), and (11), we obtain

$$(\omega_n^2 - \omega_m^2)\left[\int_0^{l_1} r_1(x)u_{1m}u_{1n} \, \mathrm{d}x + \int_0^{l_2} r_2(x)u_{2m}u_{2n} \, \mathrm{d}x\right] = 0. \tag{12}$$

Let us assume that all the eigenvalues are single roots. Then $m \neq n$ implies $\omega_m \neq \omega_n$. Under this assumption, it follows from Eq. (12) that

$$\int_0^{l_1} r_1(x)u_{1m}u_{1n} \, \mathrm{d}x + \int_0^{l_2} r_2(x)u_{2m}u_{2n} \, \mathrm{d}x = P_n\delta_{mn} \quad (n \text{ not summed}), \tag{13}$$

where

$$P_n = \int_0^{l_1} r_1(x)(u_{1n}(x))^2 \, \mathrm{d}x + \int_0^{l_2} r_2(x)(u_{2n}(x))^2 \, \mathrm{d}x. \tag{14}$$

Eq. (13) is the orthogonality condition for the two-span beam with any of the common end boundary conditions. It should be mentioned here that we followed the notational convention that is discussed just after Eq. (1) to simplify the algebra for determination of eigenfunctions.

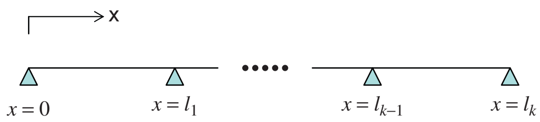


Fig. 2. Schematic view of a multi-span beam with one spatial coordinate.

In order to prove the orthogonality of the eigenfunctions for a multi-span beam, let us now consider the following two eigenvalue problems for a $k$-span beam (see Fig. 2). In the following, only one spatial coordinate $x$ in the right direction is used, and no special notational convention is used.

**Problem 1.**

$$\begin{aligned} &L_1(u_{1m}) - \omega_m^2 r_1(x) u_{1m} = 0, \qquad 0 \leqslant x \leqslant l_1, \\ &L_2(u_{2m}) - \omega_m^2 r_2(x) u_{2m} = 0, \qquad l_1 \leqslant x \leqslant l_2, \\ &\qquad \vdots \\ &L_k(u_{km}) - \omega_m^2 r_k(x) u_{km} = 0, \quad l_{k-1} \leqslant x \leqslant l_k, \end{aligned} \tag{15}$$

where

$$L_i = \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(p_i(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right) \quad (i = 1, \ldots, k), \tag{16}$$

$$p_i(x) = E_i(x) I_i(x), \quad r_i(x) = \rho_i(x) A_i(x) \quad (i = 1, \ldots, k). \tag{17}$$

The intermediate boundary conditions for $u_{1m}$ through $u_{km}$ are given by

$$\begin{aligned} &u_{1m}(l_1) = 0, \quad u_{2m}(l_1) = 0, \quad u'_{1m}(l_1) = u'_{2m}(l_1), \quad p_1(l_1)u''_{1m}(l_1) = p_2(l_1)u''_{2m}(l_1), \\ &u_{2m}(l_2) = 0, \quad u_{3m}(l_2) = 0, \quad u'_{2m}(l_1) = u'_{3m}(l_2), \quad p_2(l_2)u''_{2m}(l_2) = p_3(l_2)u''_{3m}(l_2), \\ &\qquad \vdots \\ &u_{k-1,m}(l_{k-1}) = 0, \quad u_{km}(l_{k-1}) = 0, \\ &u'_{k-1,m}(l_{k-1}) = u'_{km}(l_{k-1}), \quad p_{k-1}(l_{k-1})u''_{k-1,m}(l_{k-1}) = p_k(l_{k-1})u''_{km}(l_{k-1}). \end{aligned} \tag{18}$$

**Problem 2.**

$$\begin{aligned} &L_1(u_{1n}) - \omega_n^2 r_1(x) u_{1n} = 0, \qquad 0 \leqslant x \leqslant l_1, \\ &L_2(u_{2n}) - \omega_n^2 r_2(x) u_{2n} = 0, \qquad l_1 \leqslant x \leqslant l_2, \\ &\qquad \vdots \\ &L_k(u_{kn}) - \omega_n^2 r_k(x) u_{kn} = 0, \quad l_{k-1} \leqslant x \leqslant l_k. \end{aligned} \tag{19}$$

The intermediate boundary conditions for $u_{1n}$ through $u_{kn}$ are given by

$$\begin{aligned} &u_{1n}(l_1) = 0, \quad u_{2n}(l_1) = 0, \quad u'_{1n}(l_1) = u'_{2n}(l_1), \quad p_1(l_1)u''_{1n}(l_1) = p_2(l_1)u''_{2n}(l_1), \\ &u_{2n}(l_2) = 0, \quad u_{3n}(l_2) = 0, \quad u'_{2n}(l_2) = u'_{3n}(l_2), \quad p_2(l_2)u''_{2n}(l_2) = p_3(l_2)u''_{3n}(l_2), \\ &\qquad \vdots \\ &u_{k-1,n}(l_{k-1}) = 0, \quad u_{kn}(l_{k-1}) = 0, \\ &u'_{k-1,n}(l_{k-1}) = u'_{kn}(l_{k-1}), \quad p_{k-1}(l_{k-1})u''_{k-1,n}(l_{k-1}) = p_k(l_{k-1})u''_{kn}(l_{k-1}). \end{aligned} \tag{20}$$

It should be noted that the above two problems are the same eigenvalue problem except that the solution for each problem can be different. It should be also noted that each eigenvalue problem is a set of $k$ ordinary differential equations, which are coupled in the boundary conditions. In a completely similar manner to

Eq. (10), by using integration by parts, it can be shown from Eqs. (18) and (20) that

$$\begin{aligned}&\int_0^{l_1}[u_{1m}(p_1(x)u''_{1n})''-u_{1n}(p_1(x)u''_{1m})'']\,\mathrm{d}x+\int_{l_1}^{l_2}[u_{2m}(p_2(x)u''_{2n})''-u_{2n}(p_2(x)u''_{2m})'']\,\mathrm{d}x\\&\cdots+\int_{l_{k-1}}^{l_k}[u_{km}(p_k(x)u''_{kn})''-u_{kn}(p_k(x)u''_{km})'']\,\mathrm{d}x\\&=p_1(0)u''_{1n}(0)u'_{1m}(0)-p_1(0)u''_{1m}(0)u'_{1n}(0)-(p_1(x)u''_{1n})'|_{x=0}u_{1m}(0)+(p_1(x)u''_{1m})'|_{x=0}u_{1n}(0)\\&\quad-p_k(l_k)u''_{kn}(l_k)u'_{km}(l_k)+p_k(l_k)u''_{km}(l_k)u'_{kn}(l_k)+(p_k(x)u''_{kn})'|_{x=l_k}u_{km}(l_k)\\&\quad-(p_k(x)u''_{km})'|_{x=l_k}u_{kn}(l_k).\end{aligned}\tag{21}$$

It can be seen from Eq. (21) that all the terms associated with the intermediate support are eliminated thanks to the intermediate boundary conditions given by Eqs. (18) and (20). It should be mentioned that Eq. (21) is valid for arbitrary end boundary conditions. For any of the common end boundary conditions, the right-hand side of Eq. (21) becomes zero. Therefore, we have

$$\begin{aligned}&\int_0^{l_1}[u_{1m}(p_1(x)u''_{1n})''-u_{1n}(p_1(x)u''_{1m})'']\,\mathrm{d}x+\int_{l_1}^{l_2}[u_{2m}(p_2(x)u''_{2n})''-u_{2n}(p_2(x)u''_{2m})'']\,\mathrm{d}x\\&\cdots+\int_{l_{k-1}}^{l_k}[u_{km}(p_k(x)u''_{kn})''-u_{kn}(p_k(x)u''_{km})'']\,\mathrm{d}x=0.\end{aligned}\tag{22}$$

From Eqs. (15), (19), and (22), we obtain

$$(\omega_n^2-\omega_m^2)\left[\int_0^{l_1}r_1(x)u_{1m}u_{1n}\,\mathrm{d}x+\int_{l_1}^{l_2}r_2(x)u_{2m}u_{2n}\,\mathrm{d}x+\cdots+\int_{l_{k-1}}^{l_k}r_k(x)u_{km}u_{kn}\,\mathrm{d}x\right]=0.\tag{23}$$

Under the assumption that all the eigenvalues are single roots, it follows from Eq. (23) that

$$\int_0^{l_1}r_1(x)u_{1m}u_{1n}\,\mathrm{d}x+\int_{l_1}^{l_2}r_2(x)u_{2m}u_{2n}\,\mathrm{d}x+\cdots+\int_{l_{k-1}}^{l_k}r_k(x)u_{km}u_{kn}\,\mathrm{d}x=P_n\delta_{mn},\tag{24}$$

where no summation is taken on $n$, and

$$P_n=\int_0^{l_1}r_1(x)(u_{1n}(x))^2\,\mathrm{d}x+\int_{l_1}^{l_2}r_2(x)(u_{2n}(x))^2\,\mathrm{d}x+\cdots+\int_{l_{k-1}}^{l_k}r_k(x)(u_{kn}(x))^2\,\mathrm{d}x.\tag{25}$$

Eq. (24) is the orthogonality condition for the multi-span beam with any of the common end boundary conditions.

Hayashikawa and Watanabe [14] is one of the first papers to discuss the orthogonality of eigenfunctions of multi-span beam. Some of the more recent papers [15–17] also refer to their paper [14]. However, their result (22a) in Ref. [14] is not correct for a general $N$-span beam. Even though the system they are working on is an $N$-span continuous beam with piecewise constant cross-sectional properties, their derivation does not reflect that fact. Additionally, their evaluation of the constant for the orthogonality condition, Eq. (22b) in Ref. [14], may not be correct, unless their $N$-span beam and the end boundary conditions are of a very special kind. Since enough details on the derivation of the orthogonality are not given in Ref. [14], it is rather hard to evaluate the merit of their contribution on this subject. It certainly gives this author an impression that their derivation is not properly done even for a special case of equal mass density for all the spans. In any case their result, Eq. (22a) in Ref. [14], is only valid when the mass density $\rho_iA_i$ ($i=1\sim N$) is the same for all the spans from the first to the $n$th.

## 4. Mathematical formulation for the two-span beam vibration

In order to obtain the solution to Eqs. (1), (2) and (3), let us first consider a set of homogeneous equations, which is derived from Eq. (1)

$$\begin{aligned}E_1I_1\frac{\partial^4y_1^H}{\partial x^4}+\rho_1A_1\frac{\partial^2y_1^H}{\partial t^2}&=0\quad(0<x<l_1),\\E_2I_2\frac{\partial^4y_2^H}{\partial x^4}+\rho_2A_2\frac{\partial^2y_2^H}{\partial t^2}&=0\quad(0<x<l_2).\end{aligned}\tag{26}$$

Let us seek a solution of the following form:

$$y_1^H(x,t) = Y_1(x)\,e^{i\omega t},$$
$$y_2^H(x,t) = Y_2(x)\,e^{i\omega t}. \tag{27}$$

By substituting Eq. (27) into Eq. (26), we obtain

$$\frac{d^4 Y_1}{dx^4} - \frac{\omega^2}{a_1^2} Y_1 = 0, \quad a_1^2 = \frac{E_1 I_1}{\rho_1 A_1} \qquad (0<x<l_1),$$
$$\frac{d^4 Y_2}{dx^4} - \frac{\omega^2}{a_2^2} Y_2 = 0, \quad a_2^2 = \frac{E_2 I_2}{\rho_2 A_2} \qquad (0<x<l_2). \tag{28}$$

Even though both $Y_1$ and $Y_2$ are developed for the homogeneous equations shown in Eq. (26), they are useful for constructing the solution to the original inhomogeneous Eq. (1), as will be seen later. The boundary conditions for $Y_1$ and $Y_2$ are given by

$$Y_1(0) = 0, \quad Y_1'(0) = 0,$$
$$Y_1(l_1) = 0, \quad Y_2(l_2) = 0, \quad Y_1'(l_1) = -Y_2'(l_2), \quad E_1 I_1 Y_1''(l_1) = E_2 I_2 Y_2''(l_2),$$
$$Y_2(0) = 0, \quad Y_2''(0) = 0. \tag{29}$$

Eqs. (28) and (29) define an eigenvalue problem with eigenfunctions $Y_{1n}$ and $Y_{2n}$ and eigenvalues $\omega_n$ The eigenvalues $\omega_n$ are determined from the boundary conditions given in Eq. (29). Applying the orthogonality condition obtained in the previous section to the above system (i.e., a two-span beam with piecewise constant cross-sectional properties), we obtain

$$\rho_1 A_1 \int_0^{l_1} Y_{1m} Y_{1n}\,dx + \rho_2 A_2 \int_0^{l_2} Y_{2m} Y_{2n}\,dx = P_n \delta_{mn}. \tag{30}$$

Here $n$ is not summed and

$$P_n = \rho_1 A_1 \int_0^{l_1} Y_{1n}^2\,dx + \rho_2 A_2 \int_0^{l_2} Y_{2n}^2\,dx. \tag{31}$$

Let us expand the solution to Eq. (1) as

$$y_1(x,t) = \sum_{n=1}^{\infty} Y_{1n}(x)\phi_n(t) \quad (0<x<l_1),$$
$$y_2(x,t) = \sum_{n=1}^{\infty} Y_{2n}(x)\phi_n(t) \quad (0<x<l_2). \tag{32}$$

Substituting Eq. (32) into Eq. (1), we have

$$E_1 I_1 \sum_{n=1}^{\infty} \frac{d^4 Y_{1n}}{dx^4}\phi_n + \rho_1 A_1 \sum_{n=1}^{\infty} Y_{1n} \frac{d^2\phi_n}{dt^2} = q_1(x,t),$$
$$E_2 I_2 \sum_{n=1}^{\infty} \frac{d^4 Y_{2n}}{dx^4}\phi_n + \rho_2 A_2 \sum_{n=1}^{\infty} Y_{2n} \frac{d^2\phi_n}{dt^2} = q_2(x,t). \tag{33}$$

Let us rewrite Eq. (28) for the $n$th eigenfunction as

$$\frac{d^4 Y_{1n}}{dx^4} - \beta_{1n}^4 Y_{1n} = 0, \quad \beta_{1n}^2 = \frac{\omega_n}{a_1} \qquad (0<x<l_1),$$
$$\frac{d^4 Y_{2n}}{dx^4} - \beta_{2n}^4 Y_{2n} = 0, \quad \beta_{2n}^2 = \frac{\omega_n}{a_2} \qquad (0<x<l_2). \tag{34}$$

From Eqs. (33) and (34), we obtain

$$\sum_{n=1}^{\infty} Y_{1n}\left[\frac{\mathrm{d}^2\phi_n}{\mathrm{d}t^2}+\omega_n^2\phi_n\right]=\frac{q_1(x,t)}{\rho_1 A_1},$$
$$\sum_{n=1}^{\infty} Y_{2n}\left[\frac{\mathrm{d}^2\phi_n}{\mathrm{d}t^2}+\omega_n^2\phi_n\right]=\frac{q_2(x,t)}{\rho_2 A_2}. \tag{35}$$

From Eq. (35), we have

$$\rho_1 A_1\int_0^{l_1}\sum_{n=1}^{\infty} Y_{1m}Y_{1n}\left[\frac{\mathrm{d}^2\phi_n}{\mathrm{d}t^2}+\omega_n^2\phi_n\right]\mathrm{d}x+\rho_2 A_2\int_0^{l_2}\sum_{n=1}^{\infty} Y_{2m}Y_{2n}\left[\frac{\mathrm{d}^2\phi_n}{\mathrm{d}t^2}+\omega_n^2\phi_n\right]\mathrm{d}x$$
$$=\int_0^{l_1} Y_{1m}(x)q_1(x,t)\,\mathrm{d}x+\int_0^{l_2} Y_{2m}(x)q_2(x,t)\,\mathrm{d}x. \tag{36}$$

By using the orthogonality condition given in Eq. (30) in Eq. (36), and renaming the index from $m$ to $n$ after the algebraic manipulations, we obtain

$$\frac{\mathrm{d}^2\phi_n}{\mathrm{d}t^2}+\omega_n^2\phi_n=h_n(t), \tag{37}$$

where

$$h_n(t)=\frac{1}{P_n}\left[\int_0^{l_1} Y_{1n}(x)q_1(x,t)\,\mathrm{d}x+\int_0^{l_2} Y_{2n}(x)q_2(x,t)\,\mathrm{d}x\right] \tag{38}$$

and $P_n$ is defined in Eq. (31). It can be easily seen from Eqs. (32) and (29) that $y_1(x, t)$ and $y_2(x, t)$ satisfy the boundary conditions given in Eq. (2). Substituting Eq. (32) into Eq. (3), we have

$$\sum_{n=1}^{\infty} Y_{1n}(x)\phi_n(0)=f_1(x),\quad \sum_{n=1}^{\infty} Y_{1n}(x)\frac{\mathrm{d}\phi_n}{\mathrm{d}t}(0)=g_1(x),$$
$$\sum_{n=1}^{\infty} Y_{2n}(x)\phi_n(0)=f_2(x),\quad \sum_{n=1}^{\infty} Y_{2n}(x)\frac{\mathrm{d}\phi_n}{\mathrm{d}t}(0)=g_2(x). \tag{39}$$

From the orthogonality condition (30) and Eq. (39), we obtain

$$\phi_n(0)=\frac{1}{P_n}\left[\rho_1 A_1\int_0^{l_1} Y_{1n}(x)f_1(x)\,\mathrm{d}x+\rho_2 A_2\int_0^{l_2} Y_{2n}(x)f_2(x)\,\mathrm{d}x\right],$$
$$\frac{\mathrm{d}\phi_n}{\mathrm{d}t}(0)=\frac{1}{P_n}\left[\rho_1 A_1\int_0^{l_1} Y_{1n}(x)g_1(x)\,\mathrm{d}x+\rho_2 A_2\int_0^{l_2} Y_{2n}(x)g_2(x)\,\mathrm{d}x\right]. \tag{40}$$

It is seen from Eqs. (37) and (40) that, to determine $\phi(t)$, we need to solve the following initial value problem:

$$\frac{\mathrm{d}^2\phi_n}{\mathrm{d}t^2}+\omega_n^2\phi_n=h_n(t), \tag{41}$$

$$\phi_n(0)=c_n,$$
$$\frac{\mathrm{d}\phi_n}{\mathrm{d}t}(0)=d_n, \tag{42}$$

where

$$c_n=\frac{1}{P_n}\left[\rho_1 A_1\int_0^{l_1} Y_{1n}(x)f_1(x)\,\mathrm{d}x+\rho_2 A_2\int_0^{l_2} Y_{2n}(x)f_2(x)\,\mathrm{d}x\right],$$
$$d_n=\frac{1}{P_n}\left[\rho_1 A_1\int_0^{l_1} Y_{1n}(x)g_1(x)\,\mathrm{d}x+\rho_2 A_2\int_0^{l_2} Y_{2n}(x)g_2(x)\,\mathrm{d}x\right]. \tag{43}$$

The solution to Eq. (41) with Eq. (42) is given by

$$\phi_n(t) = \frac{d_n}{\omega_n} \sin \omega_n t + c_n \cos \omega_n t + \frac{1}{\omega_n} \int_0^t h_n(u) \sin \omega_n (t-u)\, \mathrm{d}u. \tag{44}$$

Therefore, the solution to Eq. (1) together with Eqs. (2) and (3) is given from Eq. (32) as

$$\begin{aligned} y_1(x,t) &= \sum_{n=1}^{\infty} Y_{1n}(x) \left[ \frac{d_n}{\omega_n} \sin \omega_n t + c_n \cos \omega_n t + \frac{1}{\omega_n} \int_0^t h_n(u) \sin \omega_n (t-u)\, \mathrm{d}u \right] \quad (0<x<l_1), \\ y_2(x,t) &= \sum_{n=1}^{\infty} Y_{2n}(x) \left[ \frac{d_n}{\omega_n} \sin \omega_n t + c_n \cos \omega_n t + \frac{1}{\omega_n} \int_0^t h_n(u) \sin \omega_n (t-u)\, \mathrm{d}u \right] \quad (0<x<l_2). \end{aligned} \tag{45}$$

The bending moment $M(x,t)$ can be obtained from Eq. (45) as

$$\begin{aligned} M_1(x,t) &= E_1 I_1 y_1''(x,t) \\ &= E_1 I_1 \sum_{n=1}^{\infty} \frac{\mathrm{d}^2 Y_{1n}}{\mathrm{d}x^2}(x) \left[ \frac{d_n}{\omega_n} \sin \omega_n t + c_n \cos \omega_n t + \frac{1}{\omega_n} \int_0^t h_n(u) \sin \omega_n (t-u)\, \mathrm{d}u \right] \quad (0<x<l_1), \\ M_2(x,t) &= E_2 I_2 y_2''(x,t) \\ &= E_2 I_2 \sum_{n=1}^{\infty} \frac{\mathrm{d}^2 Y_{2n}}{\mathrm{d}x^2}(x) \left[ \frac{d_n}{\omega_n} \sin \omega_n t + c_n \cos \omega_n t + \frac{1}{\omega_n} \int_0^t h_n(u) \sin \omega_n (t-u)\, \mathrm{d}u \right] \quad (0<x<l_2). \end{aligned} \tag{46}$$

## 5. Determination of the eigenfunctions

The eigenfunctions $Y_1$ and $Y_2$ are defined by Eqs. (34) and (29). The solution to Eqs. (34) and (29) is given by

$$\begin{aligned} Y_{1n}(x) &= \sinh \beta_{1n} x - \sin \beta_{1n} x - \gamma_{1n} (\cosh \beta_{1n} x - \cos \beta_{1n} x), \\ Y_{2n}(x) &= \gamma_{3n} (\gamma_{2n} \sinh \beta_{2n} x - \sin \beta_{2n} x), \end{aligned} \tag{47}$$

where

$$\begin{aligned} \gamma_{1n} &= \frac{\sinh \beta_{1n} l_1 - \sin \beta_{1n} l_1}{\cosh \beta_{1n} l_1 - \cos \beta_{1n} l_1}, \\ \gamma_{2n} &= \frac{\sin \beta_{2n} l_2}{\sinh \beta_{2n} l_2}, \\ \gamma_{3n} &= -\frac{\beta_{1n} [\cosh \beta_{1n} l_1 - \cos \beta_{1n} l_1 - \gamma_{1n} (\sinh \beta_{1n} l_1 + \sin \beta_{1n} l_1)]}{\beta_{2n} [\gamma_{2n} \cosh \beta_{2n} l_2 - \cos \beta_{2n} l_2]}, \end{aligned} \tag{48}$$

$$\begin{aligned} \beta_{1n}^2 &= \frac{\omega_n}{a_1}, \quad a_1^2 = \frac{E_1 I_1}{\rho_1 A_1}, \\ \beta_{2n}^2 &= \frac{\omega_n}{a_2}, \quad a_2^2 = \frac{E_2 I_2}{\rho_2 A_2}, \end{aligned} \tag{49}$$

and $\omega_n$ is the $n$th root of the following characteristic equation:

$$\begin{aligned} & 2E_2 I_2 \beta_{2n} (1 - \cos \beta_{1n} l_1 \cosh \beta_{1n} l_1) \sin \beta_{2n} l_2 \sinh \beta_{2n} l_2, \\ & + E_1 I_1 \beta_{1n} (\sin \beta_{1n} l_1 \cosh \beta_{1n} l_1 - \cos \beta_{1n} l_1 \sinh \beta_{1n} l_1)(\sin \beta_{2n} l_2 \cosh \beta_{2n} l_2 - \cos \beta_{2n} l_2 \sinh \beta_{2n} l_2) = 0. \end{aligned} \tag{50}$$

Since Eq. (50) is a transcendental equation for $\omega_n$, it has to be solved numerically.

## 6. Determination of the eigenfunctions when $E_1I_1 = E_2I_2$ and $\rho_1A_1 = \rho_2A_2$

Let us now consider a special case when $E_1I_1 = E_2I_2 = EI$ and $\rho_1A_1 = \rho_2A_2 = \rho A$. Then we have

$$\beta_{1n}^2 = \beta_{2n}^2 = \frac{\omega_n}{a}(\equiv \beta_n^2),$$
$$a_1^2 = a_2^2 = \frac{EI}{\rho A}(\equiv a^2). \tag{51}$$

From Eqs. (47)–(50), the eigenfunctions are given by

$$Y_{1n}(x) = \sinh \beta_n x - \sin \beta_n x - \gamma_{1n}(\cosh \beta_n x - \cos \beta_n x),$$
$$Y_{2n}(x) = \gamma_{3n}(\gamma_{2n} \sinh \beta_n x - \sin \beta_n x), \tag{52}$$

where

$$\gamma_{1n} = \frac{\sinh \beta_n l_1 - \sin \beta_n l_1}{\cosh \beta_n l_1 - \cos \beta_n l_1},$$

$$\gamma_{2n} = \frac{\sin \beta_n l_2}{\sinh \beta_n l_2}, \tag{53}$$

$$\begin{aligned}\gamma_{3n} &= -\frac{\cosh \beta_n l_1 - \cos \beta_n l_1 - \gamma_{1n}(\sinh \beta_n l_1 + \sin \beta_n l_1)}{\gamma_{2n} \cosh \beta_n l_2 - \cos \beta_n l_2} \\ &= \frac{2 \sinh \beta_n l_2(\cosh \beta_n l_1 \cos \beta_n l_1 - 1)}{(\cosh \beta_n l_1 - \cos \beta_n l_1)(\cosh \beta_n l_2 \sin \beta_n l_2 - \sinh \beta_n l_2 \cos \beta_n l_2)},\end{aligned}$$

and $\beta_n$ is defined as the $n$th root of the following characteristic equation:

$$2(1 - \cos \beta l_1 \cosh \beta l_1) \sin \beta l_2 \sinh \beta l_2 + (\sin \beta l_1 \cosh \beta l_1 - \cos \beta l_1 \sinh \beta l_1)$$
$$\times(\sin \beta l_2 \cosh \beta l_2 - \cos \beta l_2 \sinh \beta l_2) = 0. \tag{54}$$

Since Eq. (54) is also a transcendental equation (for $\beta_n$), it has to be solved numerically. It should be noted here that the eigenfunctions obtained above are different from those given in Blevins [5]. Blevins actually listed the eigenfunctions obtained by Gorman [23,24], and there was an error in Gorman's results [23,24]. In addition to the fact that Gorman used a negative of our eigenfunctions, a factor $\sinh \beta_n l_2$ was missing from their definition of $\gamma_{3n}$ above.

## 7. Application to the transient vibration of a beam with a base motion

Let us consider the case where there is a base motion to the beam but no other applied load. The governing equation for the Bernoulli–Euler beam in each span is given by

$$E_1I_1 \frac{\partial^4 y_{1\mathrm{tot}}}{\partial x^4} + \rho_1 A_1 \frac{\partial^2 y_{1\mathrm{tot}}}{\partial t^2} = 0 \quad (0<x<l_1),$$
$$E_2I_2 \frac{\partial^4 y_{2\mathrm{tot}}}{\partial x^4} + \rho_2 A_2 \frac{\partial^2 y_{2\mathrm{tot}}}{\partial t^2} = 0 \quad (0<x<l_2), \tag{55}$$

where $y_{1\text{tot}}$ and $y_{2\text{tot}}$ are the total displacement of each beam, and they are given by

$$
\begin{aligned}
y_{1\text{tot}}(x,t) &= y_1(x,t) + \left(1 - \frac{x}{l_{\text{tot}}}\right) y_{B1}(t) + \frac{x}{l_{\text{tot}}} y_{B2}(t) \quad (0<x<l_1),\\
y_{2\text{tot}}(x,t) &= y_2(x,t) + \left(1 - \frac{x}{l_{\text{tot}}}\right) y_{B2}(t) + \frac{x}{l_{\text{tot}}} y_{B1}(t) \quad (0<x<l_2),
\end{aligned} \tag{56}
$$

where

$$
l_{\text{tot}} = l_1 + l_2. \tag{57}
$$

Here $y_{B1}(t)$ and $y_{B2}(t)$ are a given displacement of the left base, and of the right base, respectively. Also in Eq. (55), as before, the spatial coordinate $(x)$ is defined in the direction to the right for the left span $(0<x<l_1)$, and it is defined in the direction to the left for the right span $(0<x<l_2)$. Substituting Eq. (56) into Eq. (55), we obtain

$$
\begin{aligned}
E_1 I_1 \frac{\partial^4 y_1}{\partial x^4} + \rho_1 A_1 \frac{\partial^2 y_1}{\partial t^2} &= q_{1B}(t) \quad (0<x<l_1),\\
E_2 I_2 \frac{\partial^4 y_2}{\partial x^4} + \rho_2 A_2 \frac{\partial^2 y_2}{\partial t^2} &= q_{2B}(t) \quad (0<x<l_2),
\end{aligned} \tag{58}
$$

where

$$
\begin{aligned}
q_{1B}(t) &= -\rho_1 A_1 \left[ \left(1 - \frac{x}{l_{\text{tot}}}\right) \frac{\partial^2 y_{B1}}{\partial t^2} + \frac{x}{l_{\text{tot}}} \frac{\partial^2 y_{B2}}{\partial t^2} \right] \quad (0<x<l_1),\\
q_{2B}(t) &= -\rho_2 A_2 \left[ \left(1 - \frac{x}{l_{\text{tot}}}\right) \frac{\partial^2 y_{B2}}{\partial t^2} + \frac{x}{l_{\text{tot}}} \frac{\partial^2 y_{B1}}{\partial t^2} \right] \quad (0<x<l_2).
\end{aligned} \tag{59}
$$

The boundary conditions for the relative displacements $y_1$ and $y_2$ are given by

$$
\begin{aligned}
&y_1(0,t) = 0, \quad y_1'(0,t) = 0,\\
&y_1(l_1,t) = 0, \quad y_2(l_2,t) = 0, \quad y_1'(l_1,t) = -y_2'(l_2,t), \quad E_1 I_1 y_1''(l_1,t) = E_2 I_2 y_2''(l_2,t),\\
&y_2(0,t) = 0, \quad y_2''(0,t) = 0.
\end{aligned} \tag{60}
$$

The initial conditions are given by

$$
\begin{aligned}
y_1(x,0) &= f_1(x), \quad \frac{\partial y_1}{\partial t}(x,0) = g_1(x) \qquad (0<x<l_1),\\
y_2(x,0) &= f_2(x), \quad \frac{\partial y_2}{\partial t}(x,0) = g_2(x) \qquad (0<x<l_2).
\end{aligned} \tag{61}
$$

Therefore, the transient vibration of a beam with a base motion is reduced to a special case of the beam vibration which was considered in the previous sections.

## 8. Transient vibration of a beam with a harmonic base motion

Let us consider the case where

$$
\begin{aligned}
E_1 I_1 &= E_2 I_2 = EI,\\
\rho_1 A_1 &= \rho_2 A_2 = \rho A,\\
y_{B1}(t) &= y_{B2}(t) = y_B(t).
\end{aligned} \tag{62}
$$

Then we have

$$
q_1(x,t) = q_2(x,t) = q(t), \tag{63}
$$

where

$$q(t) = -\rho A \frac{\partial^2 y_B}{\partial t^2}. \tag{64}$$

Let us set the initial conditions as

$$\begin{aligned} &y_1(x,0) = 0, \quad \frac{\partial y_1}{\partial t}(x,0) = 0 \qquad (0<x<l_1), \\ &y_2(x,0) = 0, \quad \frac{\partial y_2}{\partial t}(x,0) = 0 \qquad (0<x<l_2). \end{aligned} \tag{65}$$

From Eq. (45), the solution is given by

$$\begin{aligned} y_1(x,t) &= \sum_{n=1}^{\infty} \frac{Y_{1n}(x)}{\omega_n} \int_0^t h_n(u) \sin \omega_n(t-u)\, \mathrm{d}u \quad (0<x<l_1), \\ y_2(x,t) &= \sum_{n=1}^{\infty} \frac{Y_{2n}(x)}{\omega_n} \int_0^t h_n(u) \sin \omega_n(t-u)\, \mathrm{d}u \quad (0<x<l_2). \end{aligned} \tag{66}$$

Substituting Eq. (64) into Eq. (38), we have

$$h_n(t) = -\frac{\rho A}{P_n} \frac{\partial^2 y_B}{\partial t^2} \left[ \int_0^{l_1} Y_{1n}(x)\, \mathrm{d}x + \int_0^{l_2} Y_{2n}(x)\, \mathrm{d}x \right]. \tag{67}$$

Substituting Eq. (62) into Eq. (31), we have

$$P_n = \rho A \left[ \int_0^{l_1} Y_{1n}^2\, \mathrm{d}x + \int_0^{l_2} Y_{2n}^2\, \mathrm{d}x \right]. \tag{68}$$

Substituting Eq. (68) into Eq. (67), we obtain

$$h_n(t) = -L_n \frac{\partial^2 y_B}{\partial t^2}, \tag{69}$$

where

$$L_n = \frac{\int_0^{l_1} Y_{1n}(x)\, \mathrm{d}x + \int_0^{l_2} Y_{2n}(x)\, \mathrm{d}x}{\int_0^{l_1} Y_{1n}^2\, \mathrm{d}x + \int_0^{l_2} Y_{2n}^2\, \mathrm{d}x}. \tag{70}$$

Substituting Eq. (69) into Eq. (66), we obtain

$$\begin{aligned} y_1(x,t) &= -\sum_{n=1}^{\infty} \frac{L_n}{\omega_n} Y_{1n}(x) \psi_n(t) \quad (0<x<l_1), \\ y_2(x,t) &= -\sum_{n=1}^{\infty} \frac{L_n}{\omega_n} Y_{2n}(x) \psi_n(t) \quad (0<x<l_2), \end{aligned} \tag{71}$$

where

$$\psi_n(t) = \int_0^t \frac{\partial^2 y_B}{\partial u^2} \sin \omega_n(t-u)\, \mathrm{d}u. \tag{72}$$

Eq. (71) is the solution for the vibration of a beam with a general base motion when $E_1I_1 = E_2I_2 = EI$ and $\rho_1A_1 = \rho_2A_2 = \rho A$, and the initial conditions are all zero. When the base motion is harmonic, we can set

$$\frac{\partial^2 y_B}{\partial u^2} = \alpha_B \sin \omega_B u, \tag{73}$$

where $\alpha_B$ is the base acceleration. Substituting Eq. (73) into Eq. (72), and performing the integration, we obtain

$$\psi_n(t) = \frac{\alpha_B}{\omega_n^2 - \omega_B^2}(\omega_n \sin \omega_B t - \omega_B \sin \omega_n t). \tag{74}$$

Substituting Eq. (74) into Eq. (71), we finally obtain

$$\begin{aligned} y_1(x,t) &= -\alpha_B \sum_{n=1}^{\infty} \frac{L_n}{\omega_n^2 - \omega_B^2} Y_{1n}(x)\left(\sin \omega_B t - \frac{\omega_B}{\omega_n} \sin \omega_n t\right) \quad (0<x<l_1), \\ y_2(x,t) &= -\alpha_B \sum_{n=1}^{\infty} \frac{L_n}{\omega_n^2 - \omega_B^2} Y_{2n}(x)\left(\sin \omega_B t - \frac{\omega_B}{\omega_n} \sin \omega_n t\right) \quad (0<x<l_2). \end{aligned} \tag{75}$$

A non-dimensional displacement can be defined from Eq. (75) as

$$\begin{aligned} y_1^*(x,t) &= \frac{y_1(x,t)}{l} = -\frac{\alpha_B}{l} \sum_{n=1}^{\infty} \frac{L_n}{\omega_n^2 - \omega_B^2} Y_{1n}(x)\left(\sin \omega_B t - \frac{\omega_B}{\omega_n} \sin \omega_n t\right) \quad (0<x<l_1), \\ y_2^*(x,t) &= \frac{y_2(x,t)}{l} = -\frac{\alpha_B}{l} \sum_{n=1}^{\infty} \frac{L_n}{\omega_n^2 - \omega_B^2} Y_{2n}(x)\left(\sin \omega_B t - \frac{\omega_B}{\omega_n} \sin \omega_n t\right) \quad (0<x<l_2), \end{aligned} \tag{76}$$

where

$$l = \frac{l_1 + l_2}{2}. \tag{77}$$

The bending moment is obtained from Eq. (75) as

$$\begin{aligned} M_1(x,t) &= -\alpha_B EI \sum_{n=1}^{\infty} \frac{L_n}{\omega_n^2 - \omega_B^2} Y''_{1n}(x)\left(\sin \omega_B t - \frac{\omega_B}{\omega_n} \sin \omega_n t\right) \quad (0<x<l_1), \\ M_2(x,t) &= -\alpha_B EI \sum_{n=1}^{\infty} \frac{L_n}{\omega_n^2 - \omega_B^2} Y''_{2n}(x)\left(\sin \omega_B t - \frac{\omega_B}{\omega_n} \sin \omega_n t\right) \quad (0<x<l_2). \end{aligned} \tag{78}$$

A non-dimensional bending moment can be defined from Eq. (78) as

$$\begin{aligned} M_1^*(x,t) &= \frac{M_1(x,t)}{EI/l} = -\alpha_B l \sum_{n=1}^{\infty} \frac{L_n}{\omega_n^2 - \omega_B^2} Y''_{1n}(x)\left(\sin \omega_B t - \frac{\omega_B}{\omega_n} \sin \omega_n t\right) \quad (0<x<l_1), \\ M_2^*(x,t) &= \frac{M_2(x,t)}{EI/l} = -\alpha_B l \sum_{n=1}^{\infty} \frac{L_n}{\omega_n^2 - \omega_B^2} Y''_{2n}(x)\left(\sin \omega_B t - \frac{\omega_B}{\omega_n} \sin \omega_n t\right) \quad (0<x<l_2). \end{aligned} \tag{79}$$

## 9. Numerical results and discussion

Let us consider the transient vibration of a beam with a harmonic base motion with the following parameter:

$$\begin{aligned} &E_1I_1 = E_2I_2 = EI, \quad \rho_1A_1 = \rho_2A_2 = \rho A, \\ &l_1 = l_2 = l = 1\,\mathrm{m}, \\ &a = \sqrt{\frac{EI}{\rho A}} = 75\,\mathrm{m^2/s}, \\ &\omega_B = 100\,\mathrm{rad/s}, \\ &\alpha_B = 100\,\mathrm{m/s^2}. \end{aligned} \tag{80}$$

The value of "$a$" given above approximately corresponds to the steel bar of a square cross-section of $0.05\,\mathrm{m} \times 0.05\,\mathrm{m}$. If we assume that $E = 2 \times 10^{11}\,\mathrm{Pa}$, $\rho = 7800\,\mathrm{kg/m^3}$ for a square cross-section of

$0.05\,\mathrm{m} \times 0.05\,\mathrm{m}$, "$a$" actually becomes 73.0882, which is close to 75 above. The magnitude of the harmonic acceleration given above is about 10 g. The time histories of the non-dimensional beam displacement at the mid-span (left span and right span) are shown in Fig. 3. Here we have used the following notation so that we can use the conventional spatial coordinate $x$ continuously for both spans

$$y^*(x,t) = \begin{cases} y_1^*(x,t), & 0<x<l_1, \\ y_2^*(l_{\mathrm{tot}}-x,t), & l_1<x<l_{\mathrm{tot}}, \end{cases} \tag{81}$$

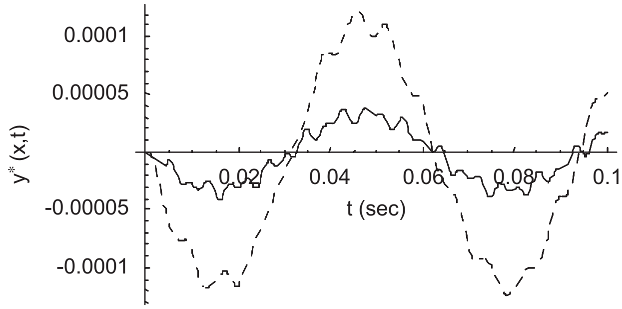


Fig. 3. Time history of the non-dimensional beam displacement at (a) $x = 0.5$ (solid line, in the left half of the beam), (b) $x = 1.5$ (dashed line, in the right half of the beam).

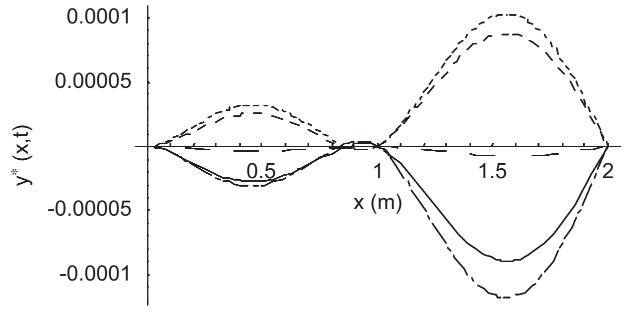


Fig. 4. Non-dimensional beam displacement over the entire span ($0<x<2$) at (a) $t = 0.01$ (solid line), (b) $t = 0.02$ (dashed-dot line), (c) $t = 0.03$ (large dashed line), (d) $t = 0.04$ (medium dashed line), and (e) $t = 0.05$ (small dashed line).

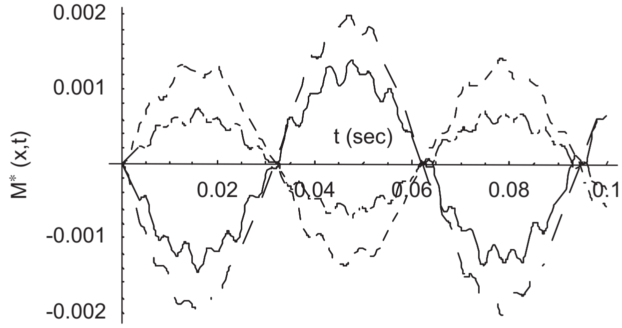


Fig. 5. Time history of the non-dimensional bending moment at (a) $x = 0$ (solid line, the clamped end), (b) $x = 0.5$ (dashed dot line, in the left half), (c) $x = 1$ (large dashed line, the mid-support), (d) $x = 1.5$ (medium dashed line, in the right half).

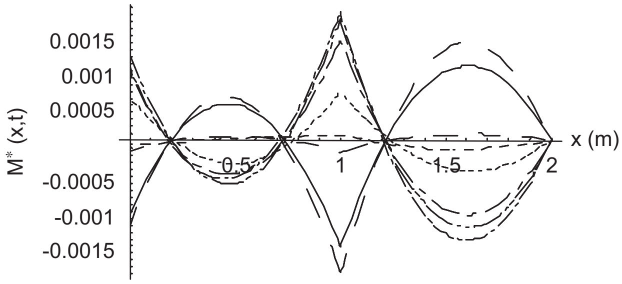


Fig. 6. Non-dimensional bending moment over the entire span ($0<x<2$) at (a) $t=0.01$ (solid line), (b) $t=0.02$ (very large dashed line), (c) $t=0.03$ (large dashed line), (d) $t=0.032$ (medium dashed line), (e) $t=0.035$ (small dashed line), (f) $t=0.04$ (large dashed dot line), (g) $t=0.0475$ (dashed dot line), and (h) $t=0.05$ (dashed double dot line).

where $l_{\text{tot}}$ is defined by Eq. (57). The non-dimensional beam displacements over the entire span at different times are shown in Fig. 4. A similar definition as Eq. (81) is also made for the non-dimensional bending moment $M^*(x,t)$. The time histories of the non-dimensional bending moment at four different locations are shown in Fig. 5. The non-dimensional bending moment diagrams over the entire span at different times are shown in Fig. 6. The numerical results in these figures are obtained by using the first 20 modes of vibration. It can be seen from the time histories that the period of the beam vibration corresponds approximately to the period of the harmonic excitation (i.e., $2\pi/100=0.0628$ s). It is seen from Fig. 4 that the beam displacement is larger in the right span as expected. It is also seen from Fig. 6 that the magnitude of the bending moment at the mid-support is often the greatest. It can also be seen from Fig. 6(c) and (d) that the inversion (reversal of the sign) of the bending moment diagram occurs near $t=0.032$ s.

## 10. Conclusion

Orthogonality of the eigenfunctions for a multi-span beam is mathematically established to the utmost generality. As an application of this orthogonality condition, forced (transient) vibration of a two-span beam is treated. The beam is modeled as a Bernoulli–Euler beam, and the boundary conditions are clamped–pinned–pinned. An exact closed-form solution is obtained for this problem. Even though there has been an enormous amount of work on beam vibration, none of the studies in the past treat an exact solution for a forced (transient) vibration of a general two-span beam with arbitrary initial conditions and arbitrary forcing functions. Therefore, to the best of the author's knowledge, the solution obtained in this paper is new. The method of solution developed in this paper establishes a general methodology for the forced (transient) vibration of a multi-span beam. The closed-form solution obtained in this paper can be used as a benchmark solution for the transient vibration of a two-span beam. Numerical results are also provided for the two-span beam vibration caused by the harmonic base excitation.

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